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Boundary layers with small departures from the Falkner–Skan profile

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The description of a laminar boundary layer with a constant pressure gradient parameter β but with an initial velocity profile close to that given by the solution of the Falkner–Skan equation for that β, is shown to lead to an eigenvalue problem in much the same manner as prevails for the Blasius solution. However, it is found that only for $\beta > \beta_0$, where β_0 corresponds to the separation value, and for the upper branch solutions are the eigenvalues all positive and the flow spatially stable. The lower branch solutions involve negative as well as positive eigenvalues and are spatially unstable.

1. Introduction

Recent work (Libby & Fox 1963; Van Dyke 1964*a*; Libby 1966) has shown that interesting and perhaps useful calculations of some laminar boundary layers with uniform external streams can be carried out by considering flows nearly described by the Blasius equation. The analysis which results from this consideration leads to an eigenvalue problem; since the associated eigenfunctions are found to form an orthogonal set, they provide means for solving a variety of problems with little numerical analysis (cf. Libby 1966). In addition, Van Dyke (1964*a*, *b*) has shown that these eigenvalues and eigenfunctions are useful in connexion with second-order boundary-layer theory and with the asymptotic approach of the boundary layer on a parabolic slab to its flat-plate behaviour far downstream of the leading edge.

It is the purpose of this paper to present the eigenvalue problem associated with the Falkner–Skan equation; thus in a physical sense we are considering flows which are nearly described by solutions to the Falkner–Skan equation, i.e. flows which are nearly similar. Previous experience has shown that the development of the eigenfunctions and their properties can be carried out in connexion with a variety of physical problems. We choose here the initial value problem, i.e. we consider flows for which the pressure gradient parameter β is constant. At some streamwise station an initial velocity profile differing somewhat from that given by the similar solution for the specified β is assumed to be given and we calculate the downstream behaviour of the boundary layer.

On physical grounds we expect the effect of the deviation of the initial profile from the similar profile to decay with increasing streamwise distance and thus for the boundary layer to approach asymptotically the similar profile. However, it will be of interest to confirm that this expectation is mathematically correct at least in some sense, i.e. in terms of a linear theory, which is exact in the limit of vanishingly small deviations from strict similarity. In this connexion we note that Serrin (1967) has recently proved for $\beta \geqslant 0$, i.e. for favourable pressure gradients, that the boundary layer with any initial profile does indeed approach asymptotically that corresponding to similarity.

We emphasize that although the present analysis will refer specifically to this initial value problem the results may be expected to be applicable to a variety of slightly non-similar boundary layers in fluids of constant properties.

The present study is restricted to the usual range of β, i.e. to $\beta_0 \leqslant \beta \leqslant 2$ where $\beta_0 = -0{\cdot}1988$. Thus we shall not consider the overshoot profiles ($f' > 1$ for some η) recently found by Libby & Liu (1967). In the range $\beta_0 < \beta < 0$ there are two solutions to the Falkner–Skan equation for each β; those with positive velocities are termed the upper branch solutions, those with regions of reverse flow the lower-branch solutions.

2. The problem specified

Consider a nonsimilar laminar boundary layer in a fluid of constant properties. The equation for the modified stream function $f(s, \eta)$ is

$$f_{\eta\eta\eta} + ff_{\eta\eta} + \beta(s)\,(1 - f_\eta^2) = 2s(f_\eta f_{\eta s} - f_s f_{\eta\eta}), \tag{2.1}$$

where the independent variables are related to the x, y co-ordinates according to

$$\eta = \rho u_e y r^j (2s)^{-\frac{1}{2}},$$
$$s = \rho\mu \int_0^x u r^{2j}\, dx$$

and where the u, v velocity components are related to $f(s, \eta)$ according to

$$\left.\begin{aligned} u &= u_e f_\eta, \\ -v &= \left\{(f + 2sf_s) + 2sf_\eta \eta \frac{d}{ds} \ln\,[u_e r^j (2s)^{-\frac{1}{2}}]\right\} \mu u_e r^j (2s)^{-\frac{1}{2}}. \end{aligned}\right\} \tag{2.2}$$

We consider a flow with $\beta(s) = \text{constant} = \beta_\infty$ and with initial and boundary conditions specified as

$$f(s_i, \eta) = F(\eta),$$
$$f(s, 0) = f_\eta(s, 0) = 0,$$
$$f_\eta(s, \infty) = 1$$

and assume a form of solution

$$f(s, \eta) = f_\infty(\eta) + f_1(s, \eta) + f_2(s, \eta) + \ldots \tag{2.3}$$

where, in accordance with the notion that the flow deviates only slightly from one described by $f_\infty(\eta)$, we consider (2.3) to involve a sequence of successively smaller functions. This clearly implies that $F(\eta)$ deviates only slightly from $f_\infty(\eta)$. Indeed in the limit of vanishingly small deviations this assumed form must be considered exact.

The zero-order solution is described by the well-known problem

$$f_\infty'''+f_\infty f_\infty''+\beta_\infty(1-f_\infty'^2) = 0 \tag{2.4}$$

subject to the conditions $f_\infty(0) = f_\infty'(0) = 0, f_\infty'(\infty) = 1$ and to the auxiliary condition that the approach of f' to unity as $\eta \to \infty$ must be exponential. Solutions to equation (2.4) may be either generated by existing numerical techniques or considered given in tabular form, e.g. in Rosenhead (1963).

The equation for $f_1(s,\eta)$ is found by substitution of (2.3) into (2.1) and by proper ordering to be

$$Lf_1 \equiv f_{1\eta\eta\eta}+f_\infty f_{1\eta\eta}-2\beta_\infty f_\infty' f_{1\eta}+f_\infty'' f_1-2s(f_\infty' f_{1s\eta}-f_\infty'' f_{1s}) = 0, \tag{2.5}$$

which is subject to $\quad f_1(s_i,\eta) = F(\eta)-f_\infty(\eta);$

$$f_1(s,0) = f_{1\eta}(s,0) = f_{1\eta}(s,\infty) = 0.$$

The equations for the successive approximations are the inhomogeneous equations

$$Lf_n = R_n, \tag{2.6}$$

where homogeneous initial and boundary conditions on f_n previal and where R_n is an explicit function of $f_1, f_2, \ldots, f_{n-1}$; for example,

$$R_2 = -f_1 f_{1\eta\eta}+\beta_\infty f_{1\eta}^2+2s(f_{1\eta}f_{1\eta s}-f_{1s}f_{1\eta\eta}).$$

The eigenvalue problem

Consider further the solution for $f_1(s,\eta)$; if a product solution $f_1(s,\eta) \sim S(s)\,N(\eta)$ is sought, it is found that $S \sim s^{-\frac{1}{2}\lambda}$ where λ is the separation constant and that there arises an eigenvalue problem for $N(\eta)$ defined by

$$N_n'''+f_\infty N_n''+(\lambda_n-2\beta_\infty)f_\infty' N_n'+(1-\lambda_n)f_\infty'' N_n = 0 \tag{2.7}$$

and $\quad N_n(0) = N_n'(0) = N_n'(\infty) = 0,$

where we put an index n on λ and $N(\eta)$ in anticipation of finding sets of eigenfunctions and related eigenvalues. If the form of solution given by (2.3) is to be consistent with the specified exponential approach of $f'(\eta)$ to unity as $\eta \to \infty$, we must restrict the approach of $N_n'(\eta)$ to zero as $\eta \to \infty$ to be exponential. It is this condition which for the Blasius solution, i.e. for $\beta_\infty = 0$, leads to discrete eigenvalues (cf. Libby & Fox 1963); we expect the same behaviour for the more general case $\beta_\infty \neq 0$ being considered here.

Properties of the eigenfunctions, $\beta > \beta_0$

A solution of (2.7) for any λ_n is $N_n \sim f_\infty'$ so that (2.7) may be reduced to a second-order equation. We must point out that for $\beta > \beta_0$, f_∞' is not an eigenfunction since it does not satisfy the homogeneous boundary condition $N_n'(0) = 0$; however, for $\beta = \beta_0, f_\infty'$ is indeed for any λ_n an eigenfunction possessing the proper exponential behaviour at $\eta \to \infty$. Accordingly, the case $\beta = \beta_0$, may be expected to be distinctive; thus it is convenient to restrict attention for the time being to $\beta > \beta_0$.

Introduce $H_n = (N_n/f'_\infty)$ as suggested by the method of variation of parameters; then (2.7) can be put in the form

$$(pH'_n)' + (q - t\lambda_n) H_n = 0, \tag{2.8}$$

where
$$p = p(\eta) = f'^3_\infty \exp\left(\int_0^\eta f_\infty d\eta\right),$$
$$t = t(\eta) = f'_\infty p,$$
$$q = q(\eta) = f'^2_\infty \exp\left(\int_0^\eta f_\infty d\eta\right) [\beta_\infty(3 - f'^2_\infty) + f_\infty f''_\infty].$$

The boundary conditions on H_n are

$$H'_n(0) - (\beta_\infty / f''_{\infty,w}) H_n(0) = H_n(\infty) = 0,$$

where again exponential decay of $H_n(\eta)$ to zero is required as $\eta \to \infty$ if $N'_n(\eta)$ is to have the proper asymptotic behaviour. Equation (2.8) is in Sturm–Liouville form (cf. e.g. Ince 1956) but our problem differs from the standard problem in some respects. In particular we are interested in the semi-infinite range of η. Moreover, for some range of β the usual restrictions on the sign and integerability of the coefficients p, t and q do not apply. Nevertheless, we follow as closely as possible standard treatments.

For $\beta \geqslant 0, f_\infty, f'_\infty > 0$ and thus p, t, q are positive for all η so that with exponential behaviour of $H_n(\eta)$ as $\eta \to \infty$ it is easy to show in the usual fashion for equations of the type of (2.8) that λ_n is real and positive and that the $H_n(\eta)$ functions are orthogonal in the sense

$$\int_0^\infty t H_n H_m d\eta = \delta_{nm} C_m, \tag{2.9}$$

where C_m is the square of the norm. For $\beta_0 < \beta < 0$ the upper branch solutions again have $f_\infty, f'_\infty > 0$ and $p, t > 0$. However, q is negative for at least some range of η so that it is not possible to prove that λ_n is positive but only that there is a minimum value of λ_n; however, the proofs of realness and of orthogonality in the sense of (2.9) carry through. We deduce from the Sturm–Liouville theory for a finite domain that for the minimum λ_n, i.e. for $\lambda_n = \lambda_1$, $H_n(\eta) = H_1(\eta)$ will have no zero for $\eta > 0$; we use this condition to identify λ_1 for the upper branch solutions and for $\beta \geqslant 0$ as well. We shall be interested in establishing for these upper branch solutions if the minimum value of λ_n, i.e. λ_1, is positive or negative. If the lowest eigenvalue is positive, then clearly all higher eigenvalues will be positive; if it is negative, then both negative and positive eigenvalues prevail.

For $\beta_0 < \beta < 0$ the lower branch solutions have f_∞, f'_∞ which are not monotonic; thus p changes sign and has a zero at the point where $f'_\infty = 0$. This appears to yield a non-standard Sturm–Liouville problem. Moreover, the demonstration of a minimum value for λ_n does not carry through; indeed the indication is that both positive and negative values of λ_n exist for this case. Nevertheless, the proof of (2.9) still applies as may be seen as follows; if $f'_\infty = 0$ at $\eta = \eta_0$, the integration from $0 \leqslant \eta < \infty$ after cross multiplication of (2.8) for n and m must be performed in the subregions $0 \leqslant \eta \leqslant \eta_0 - \epsilon$, $\eta > \eta_0 + \epsilon$. In addition a power series in $(\eta - \eta_0)$ must be used to approximate f'_∞, p, H_n, H_m, H'_n and H'_m in the neighbourhood of $\eta = \eta_0$. When this is done (2.9) and the proofs of the realness of the eigenvalues obtain.

The determination of the eigenfunctions and eigenvalues

Although there are a variety of numerical techniques which can be and have been used for the determination of the sets of $N_n(\eta)$ and related λ_n's the most accurate method is that described by Libby (1965). The basis thereof resides in the use of an outward integration of the full equation, (2.7), to an appropriate value of $\eta = \eta^*$ and an inward integration of the asymptotic form of (2.7) from

$$\eta = \eta^{**} > \eta^* \quad \text{to} \quad \eta^*.$$

Matching of the two solutions at $\eta = \eta^*$ provides an error measure for the determination of λ_n. Here we have modified somewhat this method by extending the quasilinearization technique (cf. Bellman 1963; Kalaba 1963) to an eigenvalue problem. This avoids the need to define an error measure and leads to a revised eigenvalue and revised eigenfunction in each iteration of the quasilinear scheme.

We shall not describe the details of the technique but shall present the essential relations. In particular we note that for $\eta \gg 1$, (2.7) becomes

$$N_n''' + (\eta-\kappa)\,N_n'' + (\lambda_2 - 2\beta_\infty)\,N_n' \simeq \gamma N_n(\infty)\,(\lambda_n - 1)\,(\eta-\kappa)^{-2\beta_\infty}\exp\left[-\tfrac{1}{2}(\eta-\kappa)^2\right], \tag{2.10}$$

where κ and γ are constants characterizing the asymptotic behaviour of $f_\infty(\eta)$ according to

$$f_\infty(\eta) \simeq (\eta-\kappa) + \gamma(\eta-\kappa)^{-(2\beta_\infty+2)}\exp\left[-\tfrac{1}{2}(\eta-\kappa)^2\right]\left[1-(\beta_\infty+1)(2\beta_\infty+3)(\eta-\kappa)^{-2} + O(\eta-\kappa)^2 + \ldots\right]. \tag{2.11}$$

Values of κ and γ for various values of β_∞ may be found in Rosenhead (1963); computation of additional values is a straightforward matter. Equation (2.10) must be integrated inward in order to prevent loss of accuracy due to round-off error. The initial condition to be applied at $\eta = \eta^{**}$ may be derived from the one asymptotic solution of (2.10) with the appropriate exponential decay. Whittaker & Watson (1936) provide the requisite complementary solution; the particular solution may be found by inspection. Elimination of the arbitrary constant between the solution for N_n' and that for N_n'' results in the equation

$$N_n'' \simeq -(\eta-\kappa)\,N_n'[1+(1-\lambda_n+2\beta_\infty)\,(\eta-\kappa)^{-2} + O(\eta-\kappa)^{-4}+\ldots] + (1-\lambda_n)\,\gamma N_n(\infty)\,(\eta-\kappa)^{-(2\beta_\infty-1)}\exp\left[-\tfrac{1}{2}(\eta-\kappa)^2\right][1+O(\eta-\kappa)^{-2}+\ldots]. \tag{2.12}$$

Now the second terms in the square brackets of (2.11) and (2.12) provide the means for establishing η^* and η^{**}, respectively. In addition the outward integration in the range $0 \leqslant \eta \leqslant \eta^*$ provides the estimate $N_n(\infty) \simeq N_n(\eta^*)$; thus (2.12) provides the initial condition for the inward integration in the range $\eta^* \leqslant \eta \leqslant \eta^{**}$ within a multiplicative factor. In the quasilinear scheme λ_n, $N_n(\eta^*)$ and this multiplicative factor are selected in each iteration cycle so that N_n'', N_n' and N_n are continuous at $\eta = \eta^*$.

We have found quasilinearization to work quite satisfactorily in general. There are difficulties which arise in the neighbourhood of $\beta_\infty = \beta_0$ for $\lambda_n > 0$ and whose treatment we shall discuss in more detail below. There are in addition difficulties for large negative values of λ_n such as will be seen below to arise in connexion with the lower branch solutions. These are due to the impossibility of suppressing completely the algebraic terms in the asymptotic solution of (2.10); for these cases quasilinearization involving only outward integration was found satisfactory for determining the eigenvalues.

Results of the numerical analysis

The significant results of the numerical analysis are shown in figure 1. First, we give for $\beta_\infty > 0$ and for the upper branch solutions the lowest eigenvalues, i.e. λ_1, for the range $\beta_0 < \beta_\infty \leqslant 2$; the proof that $\lambda_1 > 0$ for $\beta_\infty \geqslant 0$ is of course confirmed but in addition we see that $\lambda_1 > 0$ for the upper branch solutions. This

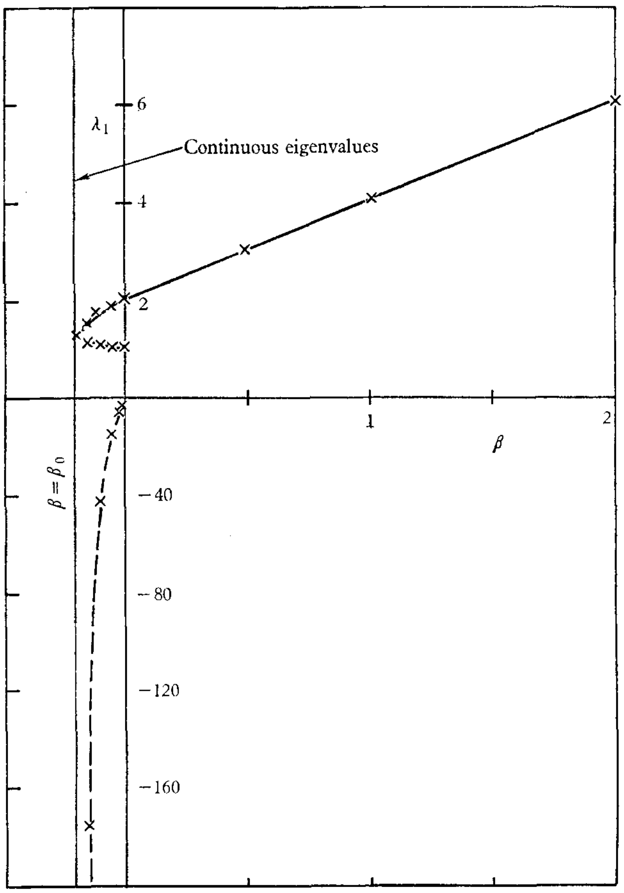

FIGURE 1. The variation of the lowest eigenvalues with the pressure gradient parameter β. ——, upper branch; – – – –, lower branch.

result confirms within this linear theory our expectation that a boundary layer with an initial profile close to the similarity profile for the prescribed β_∞ will approach asymptotically the similarity solution for increasing downstream distance. This result may be compared with Serrin (1967) for $\beta_\infty \geqslant 0$ alluded to above.

β_∞	f''_w	λ_1	C_1
2	1·6872	6·131	3·34
1	1·2326	4·177	2·63
0·5	0·9277	3·092	2·07
0	0·4696	2	1·06
−0·05	0·4000	1·877	0·893
−0·1	0·3193	1·745	0·687
−0·15	0·2164	1·594	0·420
−0·18	0·1286	1·479	0·201
−0·19883768	0	1·328	0
−0·18	−0·0977	1·215	0·499
−0·15	−0·1334	1·163	1·88
−0·1	−0·1405	1·113	6·62
−0·05	−0·1083	1·076	19·8
−0·01745	−0·0600	1·048	60·2
−0·00326	−0·0200	1·025	395
−0·00326	−0·0200	−2·235	—
−0·01745	−0·0600	−5·678	—
−0·05	−0·1083	−13·928	—
−0·1	−0·1405	−41·118	—
−0·15	−0·1334	−174·76	—

TABLE 1. The lowest eigenvalues and squares of the norms for various values of β.

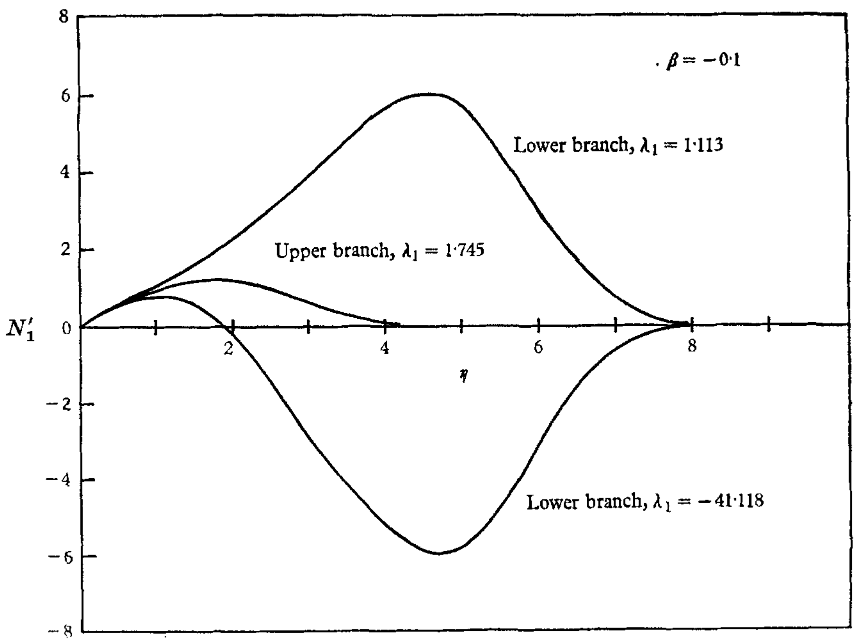

FIGURE 2. The eigenfunctions corresponding to the lowest eigenvalues for $\beta = -0·1$.

We also show in figure 1 and list in table 1 the lowest positive and highest negative eigenvalues, i.e. λ_1, for the lower branch solutions. The eigenfunctions associated with the positive eigenvalues have no zero for $N_1', \eta > 0$, but we find numerically that the first negative eigenvalue corresponds to one root for $N_1'(\eta)$. The existence of other positive and negative values for these solutions is certain; we thus conclude that the lower branch solutions are unstable in the sense that any deviation from strict similarity at an initial station will grow indefinitely with increasing downstream distance.†

n	λ_n	C_n	n	λ_n	C_n
1	3·092	2·066	11	22·406	0·230
2	4·959	1·011	12	24·365	0·215
3	6·863	0·691	13	26·326	0·203
4	8·784	0·534	14	28·288	0·192
5	10·715	0·440	15	30·251	0·182
6	12·653	0·370	16	32·216	0·174
7	14·597	0·331	17	34·183	0·167
8	16·544	0·297	18	36·150	0·160
9	18·496	0·270	19	38·118	0·154
10	20·450	0·248	20	40·088	0·149

TABLE 2. The eigenvalues and squares of the norms for $\beta_\infty = \frac{1}{2}$.

We show in figure 2 the eigenfunctions represented by $N_1'(\eta)$ for $\beta_\infty = -0{\cdot}1$, a typical negative value, corresponding to the eigenvalues shown in figure 1.

For many applications of these eigensolutions it is necessary to have a significant number of eigenvalues, e.g. to represent with reasonable care an arbitrary initial profile. To illustrate the results which might be expected we give in table 2 the first 20 eigenvalues and the square of the norms C_m for $\beta_\infty = \frac{1}{2}$.‡

The solution for $\beta_\infty > \beta_0$

The first negative eigenvalue appears to grow indefinitely as $\beta_\infty \to \beta_0^+$. Thus we conclude that the initial value problem posed above and leading in a first approximation to (2.5) is physically meaningful only for the upper branch solutions for $\beta_\infty > \beta_0$. For these cases the solution for $f_1(s, \eta)$ may be written as

$$f_1(s,\eta) = \sum_{n=1}^{\infty} A_n (s/s_i)^{-\frac{1}{2}\lambda_n} N_n(\eta), \tag{2.13}$$

where the A_n are arbitrary constants which may be selected from the initial profile by application of (2.9). There results

$$A_n = C_n^{-1} \int_0^\infty t(N_n/f_\infty')' \, [(F - f_\infty)/f_\infty']' \, d\eta. \tag{2.14}$$

† Professor K. Stewartson has informed the second author that there is a numerical study of the reversed flow, lower branch solutions that indicates that they 'flip over almost instantaneously to the normal (upper branch) form'. This would be in accord with the present findings.

‡ Tables of certain eigenfunctions and related functions of interest in applications are available from the authors upon request.

Subsequent solutions in the sequence given by (2.3) and defined by (2.6) may be found by constructing a Green's function in a manner completely analogous to that followed by Libby & Fox (1963). We have

$$f_n(s,\eta) = \int_0^\infty d\eta_0 \int_{s_i}^{s} G(s,\eta;s_0,\eta_0)\,R_n(s_0,\eta_0)\,ds_0, \tag{2.15}$$

where
$$G(s,\eta;s_0,\eta_0) = \sum_{n=1}^{\infty} D_n(\eta_0)\,N_n(\eta)\,(s_0/s)^{\frac{1}{2}\lambda_n}(2s_0)^{-1},$$

$$D_n(\eta_0) = C_n^{-1}\,[p(\eta)/f'_\infty]\,(N_n/f'_\infty)'\,|_{\eta=\eta_0}.$$

Note that there is no arbitrariness in these higher-order solutions. In general the evaluation of the $f_n(s,\eta)$ functions requires numerical evaluation of the double integrals.

The eigenfunctions in the neighbourhood of $\beta_\infty = \beta_0$

We have indicated above that the case of $\beta_\infty = \beta_0$ is special in that f'_∞ is an eigenfunction for any eigenvalue. The existence of negative eigenvalues precludes our interest in this case; however, we wish to note here that the determination of the points of tangency of the curves of λ_n versus β_∞ as $\beta_\infty \to \beta_0^+$ with this line of continuous λ_n requires a modification of the general eigenvalue problem. As suggested by the observation that for $\beta_\infty \to \beta_0^+$, $N_n \to f'_\infty/(-\beta_\infty)$ we let

$$N_n = (f'_\infty - f''_{\infty,w}\tilde{N}_n)/(-\beta_\infty). \tag{2.16}$$

Then substitution into (2.7) yields for $\tilde{N}_n(\eta)$ an equation identical with (2.7) but with revised homogeneous boundary conditions; namely, with

$$\tilde{N}_n(0) = \tilde{N}''_n(0) = \tilde{N}'_n(\infty) = 0$$

and with the normalizing condition $\tilde{N}'_n(0) = 1$.

This revised problem may be dealt with by the same numerical analysis as that described above provided the new boundary conditions are taken into account. For arbitrary β_∞, results identical with those for the original eigenvalue problem are obtained as must be the case. However, for $\beta_\infty = \beta_0$ the points of tangency such as λ_1 in figure 1 are obtained without difficulty.

We note here that (2.14) provides a further indication of the pathology of the case $\beta_\infty = \beta_0$; consider the behaviour of A_n as $\beta_\infty \to \beta_0^+$. If (2.16) is substituted into (2.14), we find that A_n is proportional to $(f''_{\infty,w})^{-1}$ times an integral whose integrand depends on $[(F-f_\infty)/f'_\infty]'$ and other well-behaved functions. Thus as $\beta_\infty \to \beta_0^+$ and $f''_{\infty,w} \to 0$, the A_n coefficients will become infinite unless $F \equiv f_\infty$, i.e. unless the deviation of the initial profile from the Falkner–Skan solution for separated flow approaches zero. This behaviour implies that the downstream length required for a deviation of the initial profile from that corresponding to similarity to decay increases as $\beta_\infty \to \beta_0^+$.

3. Concluding remarks

The initial value problem associated with flows close to those described by the Falkner–Skan is considered. It is shown that only for the pressure gradient parameter $\beta_\infty > 0$ and for the upper branch solutions for $\beta_0 < \beta_\infty \leqslant 0$ are the eigenvalues which arise in the analysis positive and thus that only for these cases is the flow spatially stable according to the linear theory on which the analysis is based. The lower branch solutions are shown to have at least one negative eigenvalue; the indication is that there are an infinite number of negative values. Thus the lower branch solutions are spatially unstable.

The eigenfunctions for the cases of applied interest may be readily computed for particular values of β_∞ and by analogy with previous experience with flows close to those described by the Blasius solution may be expected to be applicable to a variety of non-similar boundary layer problems involving arbitrary free stream pressure gradients and mass transfer.

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Note added in proof. After acceptance of this paper, the authors learned of the reference, Schönauer, W. 1965 *Z.A.M.M.* **45**, T175, in which negative eigenvalues and thus instability of the lower branch solutions are indicated. Our treatment is more complete in all respects.

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